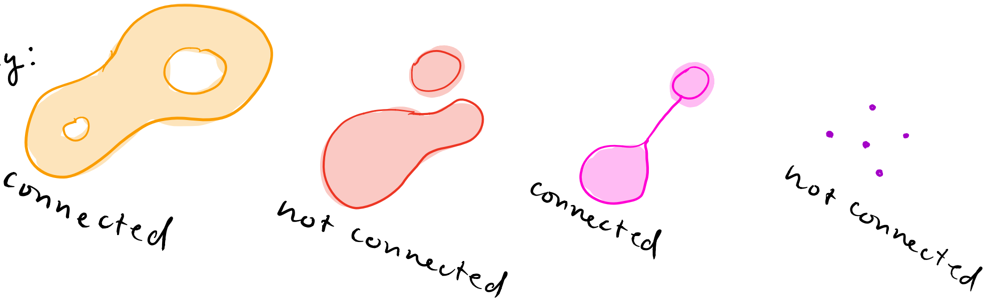
Connected spaces

Naively/Intuitively:



Def: X a topological space is <u>connected</u> if it cannot be written $X = U \cup V$ where U and V are disjoint nonempty open sets. (called a <u>separation</u> of X)

Ex: $[0,1]$ is connected (w/ standard topology):

Suppose $[0,1] = U \cup V$. WLOG $0 \in U$.

Let $a = \sup\{x \in [0,1] \mid [0,x) \subset U\}$.

U is open, so $a > 0$. If $a = 1$, we're done.

Otherwise, $a \notin U$. But then a neighborhood of a is in V so we can't have $[0,a) \subset U$.

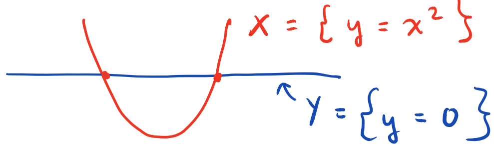
Ex: $[0,1) \cup (1,2]$ is not connected, since both $[0,1)$ and $(1,2]$ are open (in the subspace topology).

Ex: Consider $\mathbb{R}_\ell$. $\mathbb{R}_\ell = (-\infty, 0) \cup [0, \infty)$, both open, so $\mathbb{R}_\ell$ is not connected.

In fact, every subspace of $\mathbb{R}_\ell$ is disconnected other than single points. i.e. $\mathbb{R}_\ell$ is <u>totally disconnected</u>.

Note that connectedness is not preserved in subspaces:

Ex:



X is connected in $\mathbb{R}^2$ but $X \cup Y$ is not connected in Y.

However, it's preserved by continuous functions:

Thm: If $f: X \to Y$ is continuous and X connected, then $f(X)$ is connected.

Pf: Since the map $X \to f(X)$ is continuous as well, we can assume X is surjective.

Suppose $Y = U \cup V$ is a separation of Y. Then $f^{-1}(U)$ and $f^{-1}(V)$ are open, nonempty, disjoint, and their union is X. $\square$

Certain unions of connected spaces are also connected:

Thm: If $A_i \subseteq X$ are connected subspaces that all have a point in common, then $Y = \bigcup A_i$ is connected.

Pf: Suppose $Y = U \sqcup V$, U and V both open.
Suppose the common point p is in U.
$U \cap A_i$ and $V \cap A_i$ are disjoint open sets in A_i.
Since A_i is connected and $p \in A_i$, $A_i \subseteq U$ $\forall i$
$\Rightarrow Y = \bigcup A_i \subseteq U \Rightarrow Y$ is connected. $\square$

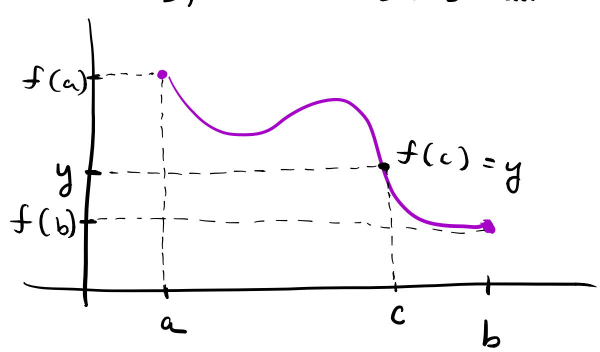
Cor: $\mathbb{R}$ is connected, as are all open, half open, and closed intervals in $\mathbb{R}$.

Pf: $[0,1]$ is connected and homeomorphic to all $[a,b]$.

$\Rightarrow \bigcup_{n\in\mathbb{N}}[-n,n] = \mathbb{R}$ is connected, since $[-n,n]$ all contain 0. $\square$

Recall the intermediate value thm from calculus:

If $f:[a,b]\longrightarrow\mathbb{R}$ is continuous, then $\forall$ y between $f(a)$ and $f(b)$, $\exists$ $c\in[a,b]$ s.t. $f(c)=y$.

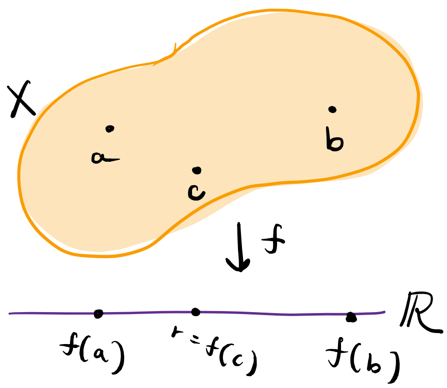


Key point: connectedness of $[a,b]$ requires the image to be connected.

More generally...

Thm: (Intermediate value theorem) Let X be a connected topological space, and $f: X\to\mathbb{R}$ continuous.

If $a, b\in X$ and r lies between $f(a)$ and $f(b)$ in r, Then $\exists\ c \in X$ s.t. $f(c) = r$.



Pf: Since X is connected, so is $f(X)$.

Consider $U = (-\infty, r) \cap f(X)$ and $V = (r, \infty) \cap f(X)$.

Both are open in $f(X)$ and nonempty, since $f(a)$ is in one, $f(b)$ in the other.

If $r \notin f(X)$, then $f(X) = U \sqcup V$, a contradiction. Thus, $r \in f(X)$.

So $\exists\ c \in X$ s.t. $f(c) = r$. $\square$

Products of connected spaces

Thm: X, Y connected $\Rightarrow$ $X \times Y$ connected.

Pf: Suppose $X \times Y = U \sqcup V$, Fix $(a, b) \in U$.

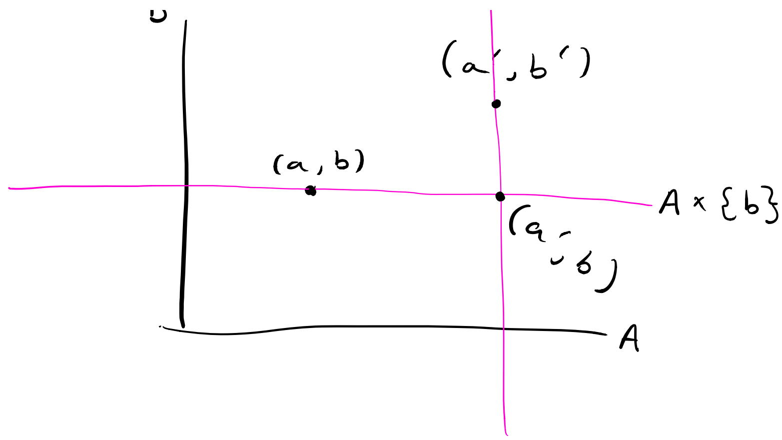
Let $(a', b') \in X \times Y$.

$\{a'\} \times B$

Then $X \times \{b\}$ is

R

connected, so
$(a', b) \in U$.



And $\{a'\} \times Y$ is
connected, so
$(a', b') \in U$.
Since (a', b') was arbitrarily chosen,
$X \times Y \subseteq U \implies X \times Y$ is connected.

Cor: Finite products of connected spaces are connected.

Pf:
$$X_1 \times X_2 \times \dots \times X_n = \underbrace{(X_1 \times X_2 \times \dots \times X_{n-1})}_{\text{connected by induction.}} \times X_n . \qquad \square$$

What about infinite products?

Claim: If $\{X_i\}_{i \in J}$ is a collection of connected spaces, then $\prod X_i$ is connected given the product topology.
(Exercise)

This is not true in the box topology...

Ex: Consider $X = \mathbb{R}^\omega$ given box topology.

Let U be the set of bounded sequences,
i.e. $(a_1, a_2, \dots)$ s.t. $\exists\ N$ s.t. $|a_i| \leq N \ \forall\ i$.

V the set of unbounded sequences.

Clearly $U \sqcup V = X$.

Why is U open?

Let $\vec{a} = (a_1, a_2, \ldots) \in X$. Consider

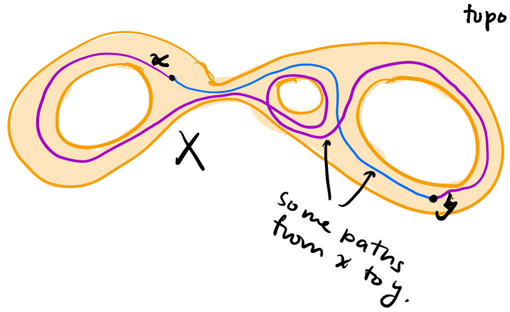
$$Y = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 - 2) \times \ldots$$

Y is open, and if $\vec{a} \in U$, $Y \subseteq U$. If $\vec{a} \in V$, $Y \subseteq V$.

Thus, U and V are both open, so the box topology is not connected.

Path connected spaces

Def: If X is a topological space and $x, y \in X$, a path from x to y is a continuous map $f : \underbrace{[a,b]}_{\text{given subspace topology in } \mathbb{R}} \longrightarrow X$ s.t. $f(a) = x$ and $f(b) = y$



X is path connected if every pair of points in X can be joined by a path.

Note: The relation $x \sim y \iff x$ and y can be connected by a path is an equivalence relation:

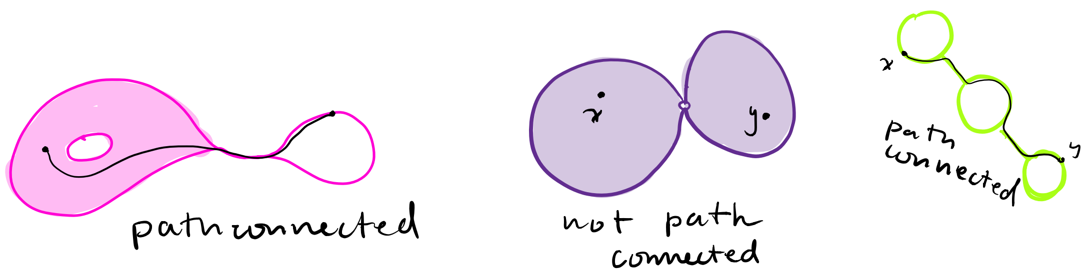
1.) $x \sim x$ by the constant path $f(t) = x$

2.) $x \sim y \Leftrightarrow y \sim x$ since we can run the path backward

3.) $x \sim y,\ y \sim z \Rightarrow x \sim z$ by running one path and then the next.

The equivalence classes are called <u>path components</u>. We'll use these a lot when we get to algebraic topology.

Ex:



How is path connectedness related to connectedness?

Thm: If X is path connected it's connected.

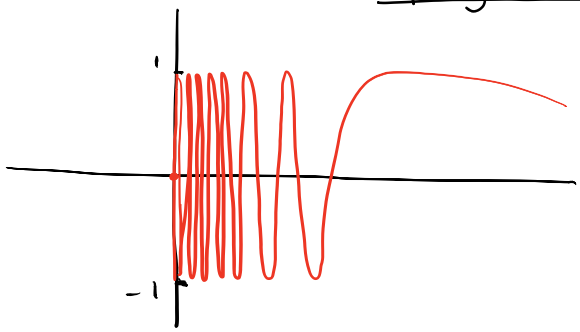
Pf: Suppose $X = U \sqcup V$, $x \in U$. Pick a path $f : [a,b] \to X$ connecting x to some other point y. Then $f([a,b])$ is connected, so $f([a,b]) \subseteq U$. Thus $y \in U$ $\forall$ $y \in X$. $\square$

The converse does not hold in general!!

Ex: Let $S \subseteq \mathbb{R}^2$ be defined:

$$S = \{(x,y) \mid \underset{A}{y = \sin(1/x)}\} \cup (0,0)$$

This is called the topologists sine curve.



A is connected, as it's the image of a connected space.

Any point $(0,0)$ is a limit point of A, since $\forall\ \varepsilon$,

There is some N s.t. $\sin(1/N) = 0$ and $N < \varepsilon$.

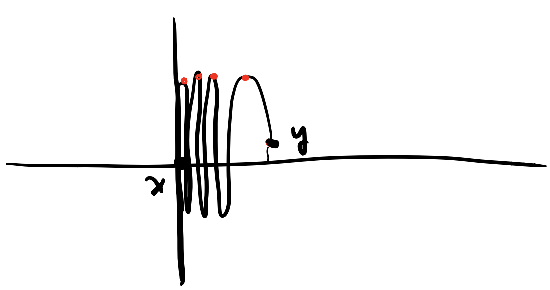
Since A is connected, and $(0,0)$ is a limit point of A, S is connected. Otherwise, if $S = U \sqcup V$, $A \subseteq U$, and if $(0,0) \in V$ then $V \cap A \neq \emptyset$. Thus, $S = U$ and S is connected.

However, S is not path connected! Consider $x = (0,0)$, $y \in A$. There is no path connecting x to y. (Exer)

Idea: Find $x_1, x_2, \ldots \in [a,b]$ s.t.

$$x_1, x_2, \ldots \to a, \text{ but}$$

$$f(x_1) = f(x_2) = \cdots = 1 \to 1 \neq 0.$$



For "well-behaved" spaces, connectedness is the same as path connectedness. e.g. manifolds.

Thm: If $U \subseteq \mathbb{R}^n$ is open, then U is connected $\iff$ path connected.

Pf: We already know "$\Leftarrow$".

"$\Rightarrow$": Suppose U is connected, $x \in U$.

Let $V \subseteq U$ be the set of points that can be reached from x by a polygonal path (i.e. a union of line segments).

V is open since for $z \in V$, any point in a ball $B_\varepsilon(z) \subseteq U$ can be reached by a straight line from z.

Claim: V is also closed in U. If $y \in \overline{V}$, then there is a ball $y \in B \subseteq U$, s.t. $B \cap V \neq \phi$. So there is a $p \in B \cap V$ that can be reached by y and by x. $\Rightarrow$ $y \in V \Rightarrow V$ is closed and open $\Rightarrow$ $V = U$. $\square$